MAU22E01 Sheet 3

Ex 1(i) The span of vectors $\bar{u}$ & $\bar{v}$ is the subspace of all vectors in $\mathbb{R}^3$ that can be written as a linear combination of $\bar{u}$ & $\bar{v}$, i.e. all vectors of the form:

$$\bar{x} = k_1 \bar{u} + k_2 \bar{v}$$

where $k_1$ & $k_2$ are free parameters.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ k_1 - 3k_2 \\ -2k_1 + 0k_2 \end{pmatrix}$$

Parametric equations for the span of $\bar{u}$ & $\bar{v}$ is therefore equal to:

$$\begin{cases} x_1 = k_1 + k_2 & \text{(A)} \\ x_2 = k_1 - 3k_2 & \text{(B)} \\ x_3 = -2k_1 & \text{(C)} \end{cases}$$

$$\Rightarrow \quad k_1 = -\frac{1}{2} x_3 \qquad \text{(C)}$$

$$x_2 = -\frac{1}{2} x_3 - 3k_2 \qquad \text{(B)} + \text{(C)}$$

$$\Rightarrow \quad 3k_2 = -\frac{1}{2} x_3 - x_2$$

$$\Rightarrow \quad k_2 = -\frac{1}{6} x_3 - \frac{1}{3} x_2 \qquad \text{(D)}$$

$$\therefore\ x_1 = -\frac{1}{2}x_3 - \frac{1}{6}x_3 - \frac{1}{3}x_2 \qquad A + C + D$$

$$\Rightarrow x_1 = -\frac{1}{3}x_2 - \frac{2}{3}x_3$$

$$\Rightarrow x_2 = -3x_1 - 2x_3$$

$$x_2 = -\frac{1}{2}x_3 + \frac{3}{6}x_3 + \frac{3}{3}x_2 \qquad B + C + D$$

$$\Rightarrow x_2 = x_2$$

The implicit equation is:

$$x_2 = -3x_1 - 2x_3$$

Ex 1ii) The span of vectors $\bar{u}, \bar{v}$ & $\bar{w}$ is the subspace of all vectors in $\mathbb{R}^4$ that can be written as a linear combination of $\bar{u}, \bar{v}$ & $\bar{w}$, i.e. all vectors of the form:

$$\bar{x} = k_1 \bar{u} + k_2 \bar{v} + k_3 \bar{w}$$

where $k_1$, $k_2$ and $k_3$ are free parameters.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ k_1 - 3k_2 + 4k_3 \\ -2k_1 - 2k_3 \\ k_2 - k_3 \end{pmatrix}$$

Parametric equations for the span of $\bar{u}, \bar{v}$ & $\bar{w}$ is thus given by:

$$\begin{cases} x_1 = k_1 + k_2 & \text{(A)} \\ x_2 = k_1 - 3k_2 + 4k_3 & \text{(B)} \\ x_3 = -2k_1 - 2k_3 & \text{(C)} \\ x_4 = k_2 - k_3 & \text{(D)} \end{cases}$$

$$k_2 = x_4 + k_3 \quad \text{(D)}$$

$$2k_1 = -x_3 - 2k_3 \quad \text{(C)}$$

$$\Rightarrow k_1 = -\frac{1}{2}x_3 - k_3$$

$$x_1 = -\frac{1}{2}x_3 - k_3 + x_4 + k_3 \quad (A) + (C) + (D)$$

$$\Rightarrow x_1 = -\frac{1}{2}x_3 + x_4 .$$

$$x_2 = -\frac{1}{2}x_3 - k_3 - 3x_4 - 3k_3 + 4k_3 \quad (B) + (C) + (D)$$

$$\Rightarrow x_2 = -\frac{1}{2}x_3 - 3x_4$$

The implicit equations are:

$$\begin{cases} x_1 = -\frac{1}{2}x_3 + x_4 \\ x_2 = -\frac{1}{2}x_3 - 3x_4 \end{cases}$$

Ex2 (i) A set of vectors are linearly dependent if the vectors satisfy a non-trivial linear relation i.e. there must be a solution to the equation:

$$k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the free parameters $k_1, k_2$ are not both equal to zero.

$$\Rightarrow \begin{cases} k_1 - k_2 = 0 \\ k_1 + 0k_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = k_2 \\ k_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \end{cases}$$

$\therefore$ There is no non-trivial linear relation, vectors $(1,1)$ & $(-1,0)$ are linearly independent.

2(ii) A set of vectors are linearly dependent if the vectors satisfy a non-trivial linear relation, i.e. there must be a solution to the equation:

$$k_1\begin{pmatrix}0\\1\\1\end{pmatrix} + k_2\begin{pmatrix}1\\1\\0\end{pmatrix} + k_3\begin{pmatrix}2\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

where the free parameters $k_1, k_2, k_3$ are not all equal to zero.

$$\Rightarrow \begin{cases} k_2 + 2k_3 = 0 \\ k_1 + k_2 = 0 \\ k_1 - 2k_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 + 2k_3 = 0 \\ k_2 = -k_1 \\ -2k_3 = -k_1 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 + 2k_3 = 0 \\ k_2 = -k_1 \\ k_3 = \frac{1}{2}k_1 \end{cases}$$

$$\Rightarrow \begin{cases} -k_1 + \frac{2}{2}k_1 = 0 \\ k_2 = -k_1 \\ k_3 = \frac{1}{2}k_1 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = -k_1 \\ k_3 = \frac{1}{2} k_1 \\ 0 = 0 \end{cases}$$

There are many non-trivial solutions. For example setting $k_1 = 1$ then $k_2 = -1$ & $k_3 = \frac{1}{2}$. $(k_1, k_2, k_3) = (1, -1, \frac{1}{2})$ is a non-trivial solution.

The vectors satisfy a non-trivial linear relation, they must be linearly dependent.

Ex 2iii) A set of vectors are linearly dependent if the vectors satisfy a non-trivial linear relation, i.e. there must be a solution to the equation:

$$k_1 \begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where the free parameters $k_1$, $k_2$ & $k_3$ are not all equal to zero.

$$\Rightarrow \begin{cases} 0k_1 + 0k_2 + 0k_3 = 0 \\ -4k_1 + 2k_2 + 2k_3 = 0 \\ 0k_1 + 3k_2 + 0k_3 = 0 \\ 0k_1 + 1k_2 + 0k_3 = 0 \\ -2k_1 + 1k_2 + 1k_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0 \\ -2k_1 + k_2 + k_3 = 0 \\ k_2 = 0 \\ k_2 = 0 \\ -2k_1 + k_2 + k_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0 \\ k_3 = 2k_1 \\ k_2 = 0 \\ k_2 = 0 \\ k_3 = 2k_1 \end{cases}$$

There are many non-trivial solutions. For example setting $k_1 = 1$ then $k_2 = 0$ and $k_3 = 2$. $(k_1, k_2, k_3) = (1, 0, 2)$ is a non-trivial solution.

The vectors satisfy a non-trivial linear relation, they must be linearly dependent.

Ex 3 (i) A set of vectors S is a basis of vector space V if the following two conditions are satisfied:

(1) The set S has the same number of vectors as the dimension of V.

(2) The vectors in the set S are linearly independent.

The set of vectors $\{(1,-1)\}$ does not form a basis of $\mathbb{R}^2$ because condition (1) is not satisfied.
The dimension of the vector space is 2 while the number of vectors in the set is 1.

Ex 3 ii) A set of vectors S is a basis of vector space V if the following two conditions are satisfied:

(1) The set S has the same number of vectors as the dimension of V.

(2) The vectors in the set S are linearly independent.

Considering the set of vectors $\{(1,0),(1,1)\}$
The dimension of the vector space $(\mathbb{R}^2)$ is 2 while the number of vectors in the set is 2.
$\Rightarrow$ condition 1 is satisfied.

- The vectors are linearly independent if they satisfy a non-trivial linear relation, i.e. there must be a solution to the equation:

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the free parameters $k_1, k_2$ are not both equal to zero.

- $$\Rightarrow \begin{cases} k_1 + k_2 = 0 \\ 0k_1 + k_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = -k_2 \\ k_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \end{cases}$$

- There is no non-trivial linear relation, vectors $(1,0)$ & $(1,1)$ are linearly independent.

Conditions (1) & (2) are satisfied.

Vectors $(1,0)$, $(1,1)$ are a basis for $\mathbb{R}^2$.

Ex 3 iii) A set of vectors $S$ is a basis of vector space $V$ if the following two conditions are satisfied:

(1) The set $S$ has the same number of vectors as the dimension of $V$.

(2) The vectors in the set $S$ are linearly independent.

Considering the set of vectors $\{(-1,-1), (2,2)\}$ The dimension of the vector space ($\mathbb{R}^2$) is 2 while the number of vectors in the set is 2.

$\Rightarrow$ condition 1 is satisfied.

The vectors are linearly independent if they satisfy a non-trivial linear relation, i.e. there must be a solution to the equation:

$$k_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the free parameters $k_1, k_2$ are not both equal to zero.

$$\Rightarrow \begin{cases} -k_1 + 2k_2 = 0 \\ -k_1 + 2k_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = \frac{1}{2} k_1 \\ k_2 = \frac{1}{2} k_1 \end{cases}$$

There are many non-trivial solutions. For example setting $k_1 = 1$ then $k_2 = \frac{1}{2}$. $(k_1, k_2) = (1, \frac{1}{2})$ is a non-trivial solution.

The vectors satisfy a non-trivial linear relation, they must be linearly dependent.

$\Rightarrow$ condition (2) is not satisfied.

Vectors $(-1,-1)$, $(2,2)$ are not a basis for $\mathbb{R}^2$.

Ex 3 iv) A set of vectors S is a basis of vector space V if the following two conditions are satisfied:
(1) The set S has the same number of vectors as the dimension of V.
(2) The vectors in the set S are linearly independent.

The set of vectors $\{(1,-1),(15,22),(-1,1)\}$ does not form a basis of $\mathbb{R}^2$. The dimension of the vector space is 2 while the number of vectors in the set is 3.

Ex 3 v) A set of vectors S is a basis of vector space V if the following two conditions are satisfied:
(1) The set S has the same number of vectors as the dimension of V.
(2) The vectors in the set S are linearly independent.

The set of vectors $\{(1,-1,2,1),(1,1,5,-3),(1,1,2,1)\}$ does not form a basis of $\mathbb{R}^4$. The dimension of the vector space is 4 while the number of vectors in the set is 3.

Ex 3 vi) A set of vectors $S$ is a basis of vector space $V$ if the following two conditions are satisfied:

(1) The set $S$ has the same number of vectors as the dimension of $V$.

(2) The vectors in the set $S$ are linearly independent.

The set of vectors $\{(1,0,1), (1,1,0), (-1,1,0)\}$ satisfy condition 1. The dimension of the vector space ($\mathbb{R}^3$) is 3 and the number of vectors in the set is 3.

The vectors are linearly independent if they satisfy a non-trivial linear relation, i.e. there must be a solution to the equation:

$$k_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where the free parameters $k_1, k_2, k_3$ are not all equal to zero.

$$\Rightarrow \begin{cases} k_1 + k_2 - k_3 = 0 \\ k_2 + k_3 = 0 \\ k_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = k_3 \\ k_2 = -k_3 \\ k_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = k_3 \\ k_3 = -k_3 \\ k_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = k_3 \\ k_3 = 0 \\ k_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = 0 \\ k_3 = 0 \\ k_1 = 0 \end{cases}$$

There is no non-trivial linear relation, vectors $(1,0,1)$, $(1,1,0)$ & $(-1,1,0)$ are linearly independent.

Conditions (1) & (2) are satisfied.

Vectors $(1,0,1)$, $(1,1,0)$ & $(-1,1,0)$ are a basis for $\mathbb{R}^3$.

Ex 4 (i) $\bar{v} = k_1 \bar{v}_1 + k_2 \bar{v}_2$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2 = k_1 + k_2 \\ 1 = -k_1 + k_2 \end{cases}$$

$$\Rightarrow \begin{cases} -2 = k_1 + k_2 \\ k_1 = k_2 - 1 \end{cases}$$

$$\Rightarrow \begin{cases} -2 = k_2 - 1 + k_2 \\ k_1 = k_2 - 1 \end{cases}$$

$$\Rightarrow \begin{cases} -1 = 2k_2 \\ k_1 = k_2 - 1 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = -1/2 \\ k_1 = k_2 - 1 \end{cases}$$

$$\Rightarrow \begin{cases} k_2 = -1/2 \\ k_1 = -3/2 \end{cases}$$

$(k_1, k_2) = (-3/2, -1/2)$ are the coordinates of $\bar{v}$ relative to the basis $\{\bar{v}_1, \bar{v}_2\}$.

Ex 4 ii) $\bar{v} = k_1 \bar{v}_1 + k_2 \bar{v}_2 + k_3 \bar{v}_3$

$$\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1 = k_1 + k_2 \\ -3 = k_1 + k_3 \\ 2 = k_2 + k_3 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 1 - k_2 \\ -3 = k_1 + k_3 \\ k_3 = 2 - k_2 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 1 - k_2 \\ -3 = 1 - k_2 + 2 - k_2 \\ k_3 = 2 - k_2 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 1 - k_2 \\ -3 = 3 - 2k_2 \\ k_3 = 2 - k_2 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = 1 - k_2 \\ k_2 = 3 \\ k_3 = 2 - k_2 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = -2 \\ k_2 = 3 \\ k_3 = -1 \end{cases}$$

$(k_1, k_2, k_3) = (-2, 3, -1)$ are the coordinates of $\bar{v}$ relative to the basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.